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# REMARKS ON THE THIN OBSTACLE PROBLEM AND CONSTRAINED GINIBRE ENSEMBLES

ARAM L. KARAKHANYAN

**ABSTRACT.** We consider the problem of constrained Ginibre ensemble with prescribed portion of eigenvalues on a given curve  $\Gamma \subset \mathbb{R}^2$  and relate it to a thin obstacle problem. The key step in the proof is the  $H^1$  estimate for the logarithmic potential of the equilibrium measure. The coincidence set has two components: one in  $\Gamma$  and another one in  $\mathbb{R}^2 \setminus \Gamma$  which are well separated. Our main result here asserts that this obstacle problem is well posed in  $H^1(\mathbb{R}^2)$  which improves previous results in  $H_{loc}^1(\mathbb{R}^2)$ .

## 1. INTRODUCTION

Let  $\Gamma$  be a regular curve in  $\mathbb{R}^2$  with locally finite length and  $\mathcal{M}_a$  the set of all probability measures such that

$$(1.1) \quad \mu(\Gamma) \geq a, \quad a \in (0, 1).$$

By an abuse of notation we let  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  be the arc-length parametrization of the curve such that

$$|\dot{\Gamma}(t)| = 1, \quad t \in \mathbb{R}.$$

In this paper we consider the minimizers of the energy

$$(1.2) \quad I[\mu] = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int Q d\mu$$

where  $Q(x)$  is a given function such that the weight function  $w = e^{-Q}$  on  $\mathbb{R}^2$  is admissible (see Definition 1.1 p.26 [8]). This means that  $w$  satisfies the following three conditions:

- (H1)  $w$  is upper semi-continuous;
- (H2)  $\{w \in \mathbb{R}^2 \text{ s.t. } w(z) > 0\}$  has positive capacity;
- (H3)  $|z|w(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ .

In higher dimensions  $\mathbb{R}^d, d \geq 3$  one can consider more general kernels

$$(1.3) \quad K(x-y) = \begin{cases} \log \frac{1}{|x-y|}, & d = 2, \\ \frac{1}{|x-y|^{d-2}}, & d \geq 3, \end{cases}$$

with  $\Gamma$  being a Lyapunov surface in  $\mathbb{R}^d$  and define the energy as follows

$$(1.4) \quad I[\mu] = \iint K(x-y) d\mu(x) d\mu(y) + \int Q d\mu.$$

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In this note we mostly confine ourselves with quadratic potentials  $Q(x) = |x|^2$  in  $\mathbb{R}^2$ , although all our results remain valid for more general  $Q$  satisfying **(H1)** – **(H3)**. Furthermore, our main result on global  $L^2$  estimate of the gradient of the equilibrium potential with kernel  $K(x - y) = |x - y|^{-d}$  remains valid in  $\mathbb{R}^d$ ,  $d \geq 3$ , see Theorem 4.1.

The functional  $I[\mu]$ , with  $Q = |x|^2$ ,  $d = 2$ , arises in the description of the convergence of the spectral measure of square  $N \times N$  matrices with complex independent, standard Gaussian entries (i.e., the Ginibre ensemble) as  $N \rightarrow \infty$ . In case when there are no constraints imposed on the eigenvalues, it is well known that the eigenvalues spread evenly in the ball of radius  $\sqrt{N}$ , and after renormalization by a factor  $\frac{1}{\sqrt{N}}$  the normalized spectral measure converges to the characteristic function of the unit disc. This is known as the circular law [4], [2]. In this context the functional  $I$  is used to prove large deviation principles for the spectral measure.

If one demands that the eigenvalues are real (i.e. when  $a = 1, \Gamma = \mathbb{R}$ ) we get the so called semicircle law. More generally, one can demand that a portion of eigenvalues is contained in a prescribed set  $\Gamma$ . This is considered in [2] when a portion of eigenvalues are contained in an open bounded subset of  $\mathbb{R}^2$  and in [4] when  $\Gamma$  is a line. These problems can be related to the thin obstacle and obstacle problems respectively. The key step in proving this is to establish  $H_{loc}^1(\mathbb{R}^2)$  estimates for the logarithmic potential

$$U^{\mu_a} = K * \mu_a$$

of the corresponding equilibrium measure. The aim of this note is to show that the thin obstacle problem is well-posed in  $H^1(\mathbb{R}^2)$  by showing that in fact  $U^{\mu_a} \in H^1(\mathbb{R}^2)$ , see Theorem 4.1. This improves the previous results in [2] and [4].

The paper is organized as follows: In the next section we prove the existence and uniqueness of the equilibrium measure  $\mu_a$  minimizing the energy  $I[\mu]$ . In section 3 we discuss some basic properties of  $\mu_a$ . In particular we show that there are two positive constants  $A_\Gamma$  and  $A_0$  such that  $2U^{\mu_a} + Q = A_\Gamma$  on  $\text{supp } \mu_a \cap \Gamma$  and  $2U^{\mu_a} + Q = A_0$  on  $\text{supp } \mu_a \setminus \Gamma$ . Furthermore,  $A_\Gamma > A_0$ . This fact will be used later to show that  $\text{supp } \mu_a \setminus \Gamma$  and  $\text{supp } \mu_a \cap \Gamma$  are disjoint.

Our main result Theorem 4.1 is contained in section 4. To prove it we study the Fourier transformations of  $U^{\mu_a}$  and  $\mu_a$ . It leads to some integral identity involving Bessel functions. This approach is based on a method of L. Carleson [3]. Finally, combining the results obtained, in section 5 we show that  $U^{\mu_a}$  solves the obstacle problem where the obstacle is given by

$$(1.5) \quad \psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases}$$

## 2. EXISTENCE OF MINIMIZERS

In this section we show the existence of a unique equilibrium measure.

**Theorem 2.1.** *Suppose  $d = 2, \Gamma \subset \mathbb{R}^2$  is a regular  $C^{1,\alpha}$  smooth planar curve without self-intersections. There is a unique minimizer  $\mu_a \in \mathcal{M}_a$  of  $I[\mu]$  such that*

$$I[\mu_a] = \inf_{\mu \in \mathcal{M}_a} I[\mu].$$

**Proof.** Observe that the uniqueness follows from the convexity of  $\mathcal{M}_a$  and can be proved as in [4]. Moreover,  $I[\mu]$  is also semicontinuous. Thus, we have to show that  $I[\mu]$  is bounded by below for all  $\mu \in \mathcal{M}_a$

and there is at least one  $\mu_0$  such that  $I[\mu]$  is finite. The lower bound follows as in the proof of Theorem 1.3 (a) p. 27 [8].

It remains show that the  $\inf_{\mu \in \mathcal{M}_a} I[\mu] < \infty$ . Let  $\chi_D$  denote the characteristic function of the set  $D$  and take

$$\mu = a \frac{1}{L} \mathcal{H}^1 \llcorner (\Gamma \cap \Omega) + (1-a) \frac{1}{|B|} \chi_B$$

where  $B = B_\rho(z) = \{x \in \mathbb{R}^2 : |x - z| < \rho\}$  with small  $\rho$  such that  $B \subset \Omega$ ,  $\Omega \subset \mathbb{R}^2$  is a compact,  $L = \mathcal{H}^1(\Gamma \cap \Omega) > 0$ , and  $\text{dist}(\Gamma, B) > 0$ . Observe that for this choice of  $\mu$  we have

$$\int_{\Omega} \log \frac{1}{|x-y|} d\mu(x) = \frac{1}{L} \int_0^L \log \frac{1}{|\Gamma(t) - y|} dt + \frac{1}{|B|} \int_B \log \frac{1}{|x-y|} d\mu(x).$$

Assuming that  $\Gamma$  is given by arc-length parametrization we have for the logarithmic energy

(2.1)

$$\mathcal{L}[\mu] = \frac{a^2}{L^2} \int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt ds + \frac{2a(1-a)}{L|B|} \int_0^L \int_B \log \frac{1}{|\Gamma(t) - y|} dt dy + \frac{(1-a)^2}{|B|^2} \int_B \int_B \log \frac{1}{|x-y|} dx dy.$$

Since  $\text{dist}(\Gamma, B) > 0$  then the second integral is bounded. As for the last integral then after change of variables  $x - y = \xi$  we have

$$\int_{B_{\rho}(z)} \log \frac{1}{|x-y|} dx = \int_{B_{\rho}(z-y)} \log \frac{1}{|\xi|} d\xi \leq \int_{B_{2\rho}(0)} \log \frac{1}{|\xi|} dx < \infty$$

where we used  $|z - y| \leq \rho$  and the fact that  $\rho$  is small by construction.

It remains to check that the first integral is finite. Let us fix  $s \in [0, L]$  Then we have that

$$\begin{aligned} \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt &= \int_{-s}^{L-s} \log \frac{1}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau = \\ &= \tau \log \frac{1}{|\Gamma(\tau + s) - \Gamma(s)|} \Big|_{-s}^{L-s} - \int_{-s}^{L-s} \tau \frac{\dot{\Gamma}(\tau + s) \cdot (\Gamma(\tau + s) - \Gamma(s))}{|\Gamma(\tau + s) - \Gamma(s)|^2} d\tau = \\ &= (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} - I_0 \end{aligned}$$

where  $I_0$  is the last integral. Using the crude estimate

$$\begin{aligned} (2.2) \quad |I_0| &\leq \int_{-s}^{L-s} |\tau| \frac{|\dot{\Gamma}(\tau + s)|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau = \int_{-s}^{L-s} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau = \\ &= \int_{[-s, L-s] \setminus (-\delta, \delta)} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau + \int_{-\delta}^{\delta} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau \\ &\leq \frac{4L^2}{C_\delta} + \int_{-\delta}^{\delta} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau \end{aligned}$$

because  $|\Gamma(\tau + s) - \Gamma(s)| \geq C_\delta$  if  $|\tau| \geq \delta$ . Finally, from  $C^{1,\alpha}$  regularity of  $\Gamma$  we get

$$\begin{aligned} (2.3) \quad |\Gamma(\tau + s) - \Gamma(s)| &= |\tau| \left| \int_0^1 \dot{\Gamma}(\sigma\tau + s) d\sigma \right| \geq \\ &\geq |\tau| \left( |\dot{\Gamma}(s)| - \int_0^1 |\dot{\Gamma}(\sigma\tau + s) - \dot{\Gamma}(s)| d\sigma \right) \\ &\geq |\tau| (1 - \delta^\alpha). \end{aligned}$$

Combining (2.3) with (2.2) we get

$$|I_0| \leq \frac{4L^2}{C_\delta} + 2\delta(1 - \delta^\alpha) < \infty.$$

Returning to the first integral in (2.1) we infer

$$\begin{aligned} \int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt ds &\leq \int_0^L \left\{ (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} + \frac{4L^2}{C_\delta} + 2\delta(1 - \delta^\alpha) \right\} ds \\ &\leq L \left[ \frac{4L^2}{C_\delta} + 2\delta(1 - \delta^\alpha) \right] + L \log \frac{1}{C_\delta} + \\ &\quad + \int_\delta^{L-\delta} \left\{ (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} \right\} ds \\ &\leq C(\delta, L) \end{aligned}$$

if we choose  $\delta > 0$  suitably small. This finishes the proof for  $d = 2$ .  $\square$

**Remark 2.2.** If  $d \geq 3$ ,  $Q(x) = |x|^2$  then clearly  $I[\mu] \geq 0$ . The upper estimate for  $I[\mu]$  follows from a similar argument if we assume that  $\Gamma$  is a Lyapunov surface and take  $\mu = a \frac{1}{L} \mathcal{H}^{d-1} \llcorner (\Gamma \cap \Omega) + (1-a) \frac{1}{|B|} \chi_B$  with  $L = \mathcal{H}^{d-1}(\Gamma \cap \Omega)$  and  $\text{dist}(B, \Gamma) > 0$ . Therefore Theorem 2.1 remains valid for  $d \geq 3$ .

### 3. BASIC PROPERTIES OF MINIMIZERS

In this section we prove some basic properties of the equilibrium measure. The arguments are along the line of those in [2]. Therefore, we mostly focus on those aspects of the proofs which are new or differ essentially. The results to follow are valid in  $\mathbb{R}^d$ ,  $d \geq 2$  unless otherwise stated.

**Lemma 3.1.** Let  $\mu_a$  be as in Theorem 2.1. Then  $\mu_a(\Gamma) = a$ .

**Proof.** If the claim fails then  $\mu_a(\Gamma) > a$ . Fix  $\delta \in (0, a)$  and let  $\mu_{a-\delta}$  be the minimizer of  $I[\cdot]$  over  $\mathcal{M}_{a-\delta} \supset \mathcal{M}_a$ . Form  $\mu = (1 - \varepsilon)\mu_a + \varepsilon\mu_{a-\delta}$ ,  $\varepsilon \in [0, 1]$ . Clearly,  $\mu \in \mathcal{M}_a$  if we choose  $\varepsilon\delta$  sufficiently small because

$$\mu(\Gamma) > a + [\mu_a(\Gamma) - a] - \varepsilon\delta.$$

Consequently, we have from the strict convexity of  $I$

$$\begin{aligned} I[(1 - \varepsilon)\mu_a + \varepsilon\mu_{a-\delta}] &< (1 - \varepsilon)I[\mu_a] + \varepsilon I[\mu_{a-\delta}] = I[\mu_a] + \varepsilon(I[\mu_{a-\delta}] - I[\mu_a]) \\ &\leq I[\mu_a] \end{aligned}$$

which is in contradiction with the fact that  $\mu_a$  is a minimizer.  $\square$

Observe that the Fréchet derivative of  $I[\mu]$  is  $2U^{\mu_a} + Q$  where

$$U^{\mu_a}(y) = \int K(x - y) d\mu_a(x).$$

It is convenient to consider variations of the equilibrium measure in terms of affine combinations. More precisely, let  $\mu_\varepsilon = (1 - \varepsilon)\mu_a + \varepsilon\nu$ ,  $\nu \in \mathcal{M}_a$ ,  $\varepsilon \in [0, 1]$ , then by direct computation we have that

$$\begin{aligned}
 (3.1) \quad I[\mu_\varepsilon] &= (1 - \varepsilon)^2 \int \int K(x - y) d\mu_a(x) d\mu_a(y) \\
 &\quad + 2\varepsilon(1 - \varepsilon) \int \int K(x - y) d\mu_a(x) d\nu(y) + \varepsilon^2 \int \int K(x - y) d\nu(x) d\nu(y) \\
 &\quad + (1 - \varepsilon) \int Q d\mu_a + \varepsilon \int Q d\nu \\
 &= I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a) \right) + O(\varepsilon^2) = \\
 &= I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q) d(\nu - \mu_a) + O(\varepsilon^2).
 \end{aligned}$$

Since  $\mu_a$  is the minimizer then  $I[\mu_a] \leq I[\mu]$ , and after sending  $\varepsilon \rightarrow 0$  it follows that

$$(3.2) \quad \int (2U^{\mu_a} + Q) d(\nu - \mu_a) \geq 0, \quad \forall \nu \in \mathcal{M}_a.$$

**Lemma 3.2.** *Let  $A_\Gamma = \frac{1}{a} \int_\Gamma (2U^{\mu_a} + Q) d\mu_a$  then quasi everywhere*

$$\begin{aligned}
 (3.3) \quad 2U^{\mu_a} + Q &= A_\Gamma \quad \text{on } \Gamma \cap \text{supp } \mu_a, \\
 &\geq A_\Gamma \quad \text{on } \Gamma.
 \end{aligned}$$

Similarly, let us denote  $A_0 = \frac{1}{1-a} \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\mu_a$  then

$$\begin{aligned}
 (3.4) \quad 2U^{\mu_a} + Q &= A_0 \quad \text{on } \text{supp } \mu_a \setminus \Gamma, \\
 &\geq A_0 \quad \text{on } \mathbb{R}^2 \setminus (\text{supp } \mu_a \setminus \Gamma).
 \end{aligned}$$

Furthermore,

$$(3.5) \quad A_\Gamma > A_0.$$

**Proof.** We first prove (3.3). Suppose that there is a set capacitable  $E$  of positive capacity such that  $\Gamma \cap E$  has zero capacity and

$$2U^{\mu_a} + Q < A_\Gamma - \delta \quad \text{q.e. on } E$$

for some positive  $\delta$ . Let  $\mu_E$  be the equilibrium measure of  $E$  and form  $\nu = \mu_a \llcorner (\mathbb{R}^2 \setminus \Gamma) + a\mu_E$ . Clearly  $\nu \in \mathcal{M}_a$ . Therefore, in view of (3.1) for the measure  $\mu_\varepsilon = \varepsilon\mu_a + (1 - \varepsilon)\nu \in \mathcal{M}_a$  we get

$$\begin{aligned}
 (3.6) \quad I[\mu_\varepsilon] &= I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a) \right) + O(\varepsilon^2) \\
 &= I[\mu_a] + \varepsilon \int_\Gamma (2U^{\mu_a} + Q) d(a\mu_E - \mu_a) + O(\varepsilon^2) \\
 &= I[\mu_a] + \varepsilon \left( a \int_\Gamma (2U^{\mu_a} + Q) d\mu_E - aA_\Gamma \right) + O(\varepsilon^2) \\
 &< I[\mu_a] - a\varepsilon\delta + O(\varepsilon^2) \\
 &< I[\mu_a]
 \end{aligned}$$

if  $\varepsilon$  and  $\delta$  are sufficiently small. This will be in contradiction with the fact that  $\mu_a$  is the minimizer. Thus we have proved that  $2U^{\mu_a} + Q \geq A_\Gamma$  q.e. on  $\Gamma$ .

Next we show that on  $\text{supp } \mu_a \cap \Gamma$  we have  $2U^{\mu_a} + Q = A_\Gamma$  q.e. Indeed, from the definition of  $A_\Gamma$  it follows

$$aA_\Gamma = \int_{\Gamma} (2U^{\mu_a} + Q) d\mu_a \geq aA_\Gamma$$

where the last inequality follows from the first inequality in (3.3). The proof of (3.4) is similar. In order to prove the last claim  $A_\Gamma > A_0$  we first observe that there exists a measure  $\nu \in \mathcal{M}_a$  such that

- $a > \nu(\Gamma)$ ,
- $I[\nu] \leq I[\mu_a]$ .

First notice that  $\mathcal{M}_a \subset \mathcal{M}_{a-\delta}$  for  $\delta \in (0, a)$ . Fix such  $\delta > 0$  and let  $\mu_{a-\delta}$  be the minimizer of  $I[\cdot]$  over  $\mathcal{M}_{a-\delta}$ . Then by Lemma 3.1  $\mu_{a-\delta}(\Gamma) = a - \delta < a$  and  $I[\mu_{a-\delta}] = \inf_{\mathcal{M}_{a-\delta}} I[\mu] \leq I[\mu_a] = \inf_{\mathcal{M}_a} I[\mu]$ . Therefore one can take  $\nu = \mu_{a-\delta}$ .

From the strict convexity of  $I$  it follows that

$$I[\nu] > I[\mu_a] + \langle DI[\mu_a], \nu - \mu_a \rangle$$

where  $DI[\mu] = 2U^\mu + Q$  is the Fréchet derivative of  $I[\mu]$ . Therefore, from the properties of  $\nu$  we infer

$$(3.7) \quad 0 \geq I[\nu] - I[\mu_a] > \langle DI[\mu_a], \nu - \mu_a \rangle$$

or equivalently

$$\langle 2U^{\mu_a} + Q, \nu - \mu_a \rangle < 0.$$

On the other hand

$$(3.8) \quad \int (2U^{\mu_a} + Q) d\mu_a = aA_\Gamma + (1-a)A_0$$

while

$$\int (2U^{\mu_a} + Q) d\nu = \int_{\Gamma} (2U^{\mu_a} + Q) d\nu + \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\nu \geq \nu(\Gamma)A_\Gamma + \nu(\mathbb{R}^2 \setminus \Gamma)A_0.$$

This together with (3.8), (3.7) yields

$$aA_\Gamma + (1-a)A_0 > \nu(\Gamma)A_\Gamma + (1-\nu(\Gamma))A_0 \Rightarrow A_0(\nu(\Gamma) - a) > A_\Gamma(\nu(\Gamma) - a).$$

Finally, the property  $\nu(\Gamma) < a$  implies that  $A_\Gamma > A_0$ . □

**Corollary 3.3.**  *$\text{supp } \mu_a$  is compact.*

**Proof.** If  $d \geq 3$  then  $K(x - y) \geq 0$ , hence by Lemma 3.2 for  $x \in \text{supp } \mu_a$  we have

$$(3.9) \quad \max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) \rightarrow \infty \quad \text{if } |x| \rightarrow \infty$$

which is a contradiction. If  $d = 2$  then from the triangle inequality we get that

$$(3.10) \quad K(x - y) \geq -\log|x| - \log\left(1 + \frac{|y|}{|x|}\right).$$

Consequently, for  $x \in \text{supp } \mu_a$

$$\begin{aligned} \max(A_\Gamma, A_0) &\geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) - 2\log|x| - \int \log\left(1 + \frac{|y|}{|x|}\right) d\mu_a \\ &= Q(x) - 2\log|x| + O(1) \rightarrow \infty \quad \text{if } |x| \rightarrow \infty \end{aligned}$$

for sufficiently large  $|x|$ , where the last inequality follows from (4.12) and  $\int Q d\mu_a < I[\mu_a] < \infty$ . Since  $Q = |x|^2$  (or for the general case from the hypotheses on  $Q$  (H1) – (H3)) it again follows that  $\text{supp } \mu_a$  is bounded.  $\square$

#### 4. GLOBAL $L^2$ ESTIMATES FOR $U^{\mu_a}$ AND $\nabla U^{\mu_a}$

Our main result is contained in the following

**Theorem 4.1.** *Let  $U^{\mu_a}(y) = \int K(x - y) d\mu_a$ , if  $d \geq 3$  then  $\nabla U^{\mu_a} \in L^2(\mathbb{R}^d)$ . If  $d = 2$  then  $U^{\mu_a} \in H^1(\mathbb{R}^2)$ . Furthermore, there holds*

$$(4.1) \quad \|U^{\mu_a}\|_{H^1(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].$$

Here  $\mathcal{E}[\mu]$  is the energy of  $\mu$  defined as  $\int \int K(x - y) d\mu(x) d\mu(y)$ .

**Remark 4.2.** *It is shown in [3] that  $\mathcal{E}[\mu] > 0$  for any probability measure  $\mu$  and  $d \geq 2$ . In fact, this can be seen from the proof to follow (see also Corollary 4.3).*

**Proof.** The case  $d \geq 3$  follows from Lemma 1.6 p. 92 [7] (see also Lemma 17 p. 95), which assert that

$$\frac{\partial U^{\mu_a}(x)}{\partial x_i} = \int \frac{\partial K(x - y)}{\partial x_i} d\mu_a$$

almost everywhere and moreover

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^d} |\nabla U^{\mu_a}|^2 \leq \int \int K(x - y) d\mu_a(x) d\mu_a(y) = \mathcal{E}[\mu_a].$$

The case of the logarithmic potential follows from a modification of the argument by L. Carleson [3] Lemma 3 page 22. We begin with computing the Fourier transformation of  $K$ . Note that since  $\text{supp } \mu_a$  is compact we can assume that  $K(r) = 0$  for  $r \geq r_0$  for some fixed  $r_0 > 0$ . We have

$$\begin{aligned} \widehat{K}(\xi) &= \int K(x) e^{-2\pi i \langle x, \xi \rangle} dx = \int K(x) e^{-2\pi i \langle x, |\xi|, \frac{\xi}{|\xi|} \rangle} dx \\ &= \frac{1}{4\pi^2 |\xi|^2} \int K\left(\frac{y}{2\pi |\xi|}\right) e^{i \langle y, \frac{\xi}{|\xi|} \rangle} dy. \end{aligned}$$

Let us denote  $K_0(y) = K\left(\frac{y}{2\pi |\xi|}\right)$  and define

$$F(\eta) = \int K_0(y) e^{i\pi \langle y, \eta \rangle} dy, \quad \eta = \frac{\xi}{|\xi|}.$$

From Lemma 2 p. 21 [3] it follows that there is a universal constant  $c_1$  such that

$$F(\eta) = c_1 \int_0^\infty K_0(r) J(r) r dr, \quad |\eta| = 1$$

where  $J$  is the Bessel function

$$(4.2) \quad J(r) = -J''(r) - \frac{J'(r)}{r}, \quad J(0) = 1, J'(0) = 0, \quad J(r) < 1, r \neq 0.$$



Therefore  $F(\eta)$  can be further simplified as follows

$$(4.3) \quad \begin{aligned} F(\eta) &= -c_1 \int_0^\infty K_0(r)(rJ(r))' dr = \\ &= c_1 \int_0^{2\pi|\xi|r_0} rJ'(r)K_0'(r)dr \end{aligned}$$

because from the definition of  $K_0$  we have  $\text{supp } K_0 \subset [0, 2\pi|\xi|r_0]$ . Moreover,  $K_0'(r) = -\frac{1}{r}$  hence

$$(4.4) \quad F(\eta) = c_1(1 - J(2\pi|\xi|r_0)).$$

Consequently,

$$(4.5) \quad \widehat{K}(\xi) = \frac{c_1}{4\pi^2|\xi|^2}(1 - J(2\pi|\xi|r_0)).$$

Next we restrict  $\mu_1 = \mu_a \lfloor \mathcal{C}$  where  $\mathcal{C} \subset \text{supp } \mu_a$  is a compact such that  $U^{\mu_1}$  is continuous. Observe that  $\int U^{\mu_a} d\mu_a$  is finite hence  $U^{\mu_a}$  is finite  $\mu_a$  almost everywhere. By Theorem 1.8 p. 70 [7] for every  $\varepsilon > 0$  small there is a restriction of  $\mu_a$  such that

$$0 \leq \int \mu_a - \int \mu_1 < \varepsilon.$$

Note that if  $\tau = \mu_a - \mu_1$  then we have

$$|\mathcal{E}[\mu_a] - \mathcal{E}[\mu_1]| = \left| \int U^{\mu_a - \mu_1} d\mu_a + \int U^{\mu_a - \mu_1} d\mu_1 \right| = \left| \int (U^{\mu_a} + U^{\mu_1}) d\tau \right| = O(\varepsilon).$$

Let  $\phi_n(y) = n^{\frac{d}{2}} e^{-n\pi|y|^2}$  be the sequence of normalised Gaussian kernels. It is well-known that  $\phi_n$  is a mollification kernel for every  $n \in \mathbb{N}$  and moreover  $\widehat{\phi_n} = e^{-\frac{\phi|\xi|^2}{n}}$ . From the Parseval relation

$$(4.6) \quad \int (\phi_n * U^{\mu_1}) d\mu_1 = \int \widehat{\phi_n} \widehat{K} |\widehat{\mu_1}|^2.$$

If we first send  $n \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$  to conclude the identity

$$(4.7) \quad \mathcal{E}[\mu_a] = \int \widehat{K} |\widehat{\mu_a}|^2.$$

On the other hand  $\widehat{U^{\mu_a}} = \widehat{K} \widehat{\mu_a}$ , which yields

$$(4.8) \quad \begin{aligned} \mathcal{E}[\mu_a] &= \int \widehat{K}(\xi) \frac{|\widehat{U^{\mu_a}}(\xi)|^2}{|\widehat{K}(\xi)|^2} d\xi \\ &= \int \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} |\widehat{U^{\mu_a}}(\xi)|^2 d\xi \\ &= \int_{|\xi| < \delta} + \int_{|\xi| \geq \delta}. \end{aligned}$$

Using the expansion  $J(t) = \sum_{s=0}^\infty \frac{(-1)^s}{(s!)^2} \left(\frac{t}{2}\right)^{2s} = 1 - \frac{t^2}{4} + \frac{t^4}{64} + \dots$  we see that

$$\frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} = \frac{1}{r_0^2 c_1} \frac{4}{\left(1 - \frac{(2\pi r_0|\xi|)^2}{16} + \dots\right)}$$

hence the first integral is bounded below by  $C(\delta)\frac{1}{r_0^2 c_1} \int_{|\xi| < \delta} |\widehat{U^{\mu_a}}(\xi)|^2 d\xi$  for sufficiently small  $\delta > 0$ . As for the second integral, we have

$$(4.9) \quad \int_{|\xi| \geq \delta} \frac{4\pi^2 |\xi|^2}{c_1(1 - J(2\pi r_0 |\xi|))} |\widehat{U^{\mu_a}}(\xi)|^2 d\xi \geq \frac{4\pi^2 \delta^2}{c_1} \int_{|\xi| \geq \delta} |\widehat{U^{\mu_a}}(\xi)|^2 d\xi.$$

Combining we see that  $\widehat{U^{\mu_a}} \in L^2(\mathbb{R}^2)$  which, after we apply Parseval's relation again, yields  $U^{\mu_a} \in L^2(\mathbb{R}^2)$  and

$$(4.10) \quad \|U^{\mu_a}\|_{L^2(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].$$

To finish the proof we use that  $4\pi^2 |\xi|^2 |\widehat{U^{\mu_a}}|^2 = |\widehat{\nabla U^{\mu_a}}|^2$  which together with (4.8) implies that

$$(4.11) \quad \mathcal{E}[\mu_a] = \int \frac{1}{c_1(1 - J(2\pi r_0 |\xi|))} |\widehat{\nabla U^{\mu_a}}(\xi)|^2 d\xi \geq \frac{1}{c_1} \int |\widehat{\nabla U^{\mu_a}}(\xi)|^2 d\xi$$

which finishes the proof.  $\square$

**Corollary 4.3.** *Let  $\mu_a$  be as in Theorem 2.1. Then there holds*

$$(4.12) \quad \mathcal{E}[\mu_a] = \int U^{\mu_a} d\mu_a > 0.$$

## 5. THE THIN OBSTACLE PROBLEM

From the  $H^1(\mathbb{R}^2)$  estimate for  $U^{\mu_a}$  it follows that  $U^{\mu_a}$  is a solution to some variational inequality, and hence  $U^{\mu_a}$  can be interpreted as a solution to an obstacle problem with a combination of both thin (on  $\Gamma$ ) and "thick" obstacles (on  $\mathbb{R}^2 \setminus \Gamma$ ). It is convenient to define the obstacle as follows

$$(5.1) \quad \psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases}$$

**Lemma 5.1.** *Let  $U^{\mu_a}$  be the logarithmic potential of  $\mu_a$  and define*

$$\mathcal{K} = \{v \in H_{loc}^1(\mathbb{R}^2) \text{ s.t. } v - U^{\mu_a} \text{ has bounded support in } \mathbb{R}^2, v \geq \psi\}.$$

*Then  $U^{\mu_a}$  solves the following obstacle problem:*

$$\int \nabla U^{\mu_a} \nabla (v - U^{\mu_a}) \geq 0, \quad \forall v \in \mathcal{K}.$$

The proof is the same as in [2].

**Corollary 5.2.**  $\text{dist}(\Gamma, \text{supp}(\mu_a \setminus \Gamma)) > 0$ .

**Proof.** This follows from the estimate  $A_\Gamma > A_0$ . Indeed, let us assume that  $x_0 \in \Gamma \cap \text{supp} \mu_a$  and there is a sequence  $\{x_k\}_{k=1}^\infty, x_k \in \text{supp} \mu_a \setminus \Gamma$  such that  $\lim_{k \rightarrow \infty} x_k \rightarrow x_0$ . Using the lower semicontinuity of  $U^{\mu_a}$  (see Lemma 1 p.15 [3]) we see that

$$(5.2) \quad \frac{1}{2}(A_0 - |x_0|^2) = \liminf_{x_k \rightarrow x_0} U^{\mu_a}(x_k) \geq U^{\mu_a}(x_0).$$

Let  $\rho > 0$  be such that  $\{x_k\} \subset B_\rho(x_0)$ . If  $\rho$  is small then  $\Gamma$  divides  $B_\rho(x_0)$  into two parts  $D^+$  and  $D^-$ . To fix the ideas let us suppose that  $D^+$  contains a subsequence  $\{x_k\}$ . Let  $h$  be the harmonic function in  $D^+$

such that  $h = \psi$  on  $\partial D^+$ . Observe that  $h$  is continuous at  $x_0$  because  $\Gamma \in C^{1,\alpha}$ . Since  $U^{\mu_a}$  is superharmonic and on  $\partial D^+$  we have  $U^{\mu_a} \geq \psi = h$  then the comparison principle implies that

$$(5.3) \quad U^{\mu_a}(x_0) \geq h(x_0) = \frac{1}{2}(A_\Gamma - |x_0|^2).$$

Combining (5.2) and (5.3) we see that  $A_0 \geq A_\Gamma$  which is a contradiction in view of (3.5).  $\square$

From Corollary 5.2 it follows that near  $\Gamma$  the potential  $U^{\mu_a}$  is a solution to a thin obstacle problem in the following sense, see [5] p. 108:

$$(5.4) \quad \left. \begin{aligned} U^{\mu_a} &\geq \frac{1}{2}(A_\Gamma - Q) \\ \frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} &\geq 0 \\ (u - \frac{1}{2}(A_\Gamma - Q)) \left( \frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} \right) &= 0 \end{aligned} \right\} \text{ on } \Gamma$$

where  $n^\pm$  are the outward normals on the  $\Gamma$  corresponding to the domains that  $\Gamma$  separates. In particular, if  $\Gamma$  is  $C^3$  regular then  $U^{\mu_a}$  is  $C^{1,\alpha}$  up to  $\Gamma$  from each of its side, see Theorem 11.4 p.111 [5].

A particular case is  $\Gamma = \mathbb{R}$  [4]. Using a simple symmetrization argument (see e.g. [6] p. 119 Theorem 4.6) we can show that the potential  $U^{\mu_a}$  is symmetric w.r.t. the real line and hence we get the Signorini problem near  $\mathbb{R}$  [5] p. 111.

One can make the connections with the obstacle problem more explicit by using the  $H^1(\mathbb{R}^2)$  estimate in Theorem 4.1 and transforming the energy  $I[\mu_a]$ . Let  $R > 0$  be fixed then using the divergence theorem

$$(5.5) \quad \begin{aligned} \int_{B_R} U^{\mu_a} d\mu_a &= -\frac{1}{2\pi} \int_{B_R} U^{\mu_a} \Delta U^{\mu_a} = \\ &= \frac{1}{2\pi} \int_{B_R} |\nabla U^{\mu_a}|^2 - \frac{1}{2\pi} \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a}. \end{aligned}$$

For a.e.  $R > 0$  the last integral can be estimated as follows

$$\left| \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a} \right| \leq \int_{\partial B_R} |U^{\mu_a}| |\nabla U^{\mu_a}| \leq \int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2.$$

From Theorem 4.1 and Fubini's theorem it follows that

$$\int_{\mathbb{R}^2} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) = \int_0^\infty \int_{\partial B_R} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) dR.$$

Consequently,

$$\int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2 \rightarrow 0 \quad R \rightarrow \infty$$

and we infer from (5.5) that

$$\int_{\mathbb{R}^2} U^{\mu_a} d\mu_a = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2.$$

Recalling that by Corollary 3.3  $\text{supp } \mu_a \subset B_{r_0}$  for some  $r_0 > 0$  and using the divergence theorem again we conclude

$$(5.6) \quad \begin{aligned} \int_{B_{r_0}} |x|^2 d\mu_a &= -\frac{1}{2\pi} \int_{B_{r_0}} |x|^2 \Delta U^{\mu_a} = -\frac{1}{2\pi} \int_{B_{r_0}} U^{\mu_a} \Delta |x|^2 + \frac{1}{2\pi} \int_{\partial B_{r_0}} (2r_0 U^{\mu_a} - r_0^2 \partial_n U^{\mu_a}) \\ &= -\frac{2}{\pi} \int_{B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^{\mu_a} + r_0^2. \end{aligned}$$

Combining these we have that the energy can be rewritten in terms of  $U^{\mu_a}$  in the following form

$$I[\mu_a] = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2 - \frac{2}{\pi} \int_{B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^{\mu_a} + r_0^2.$$

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